Spherical Polar Coordinates & Unit Vectors

- On HW 2, you should have derived the following expressions for the unit vectors $\hat{r}$, $\hat{\theta}$, & $\hat{\phi}$:

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

- This makes it easy to write the $r, \theta,$ & ϕ components of a vector if you already know its $x, y,$ & z components in Cartesian coords.

- Like we said in class, the components of a vector $\vec{A}$ in SPC are:

$$A_r = \vec{A}\cdot\hat{r} \qquad A_\theta = \vec{A}\cdot\hat{\theta} \qquad A_\phi = \vec{A}\cdot\hat{\phi}$$

So

$\hat{x}\cdot\hat{r} = \sin\theta\cos\phi$] The r-component of $\hat{x}$

$\hat{x}\cdot\hat{\theta} = \cos\theta\cos\phi$] The θ-component of $\hat{x}$

$\hat{x}\cdot\hat{\phi} = -\sin\phi$] The ϕ-component of $\hat{x}$

$$\hookrightarrow \hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$
$$\hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$
$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

Work these out the same way we worked out the components of $\hat{x}$

- We can also figure out how to write the position vector in SPC this way

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$
$$= r\sin\theta\cos\phi\;\hat{x} + r\sin\theta\sin\phi\;\hat{y} + r\cos\theta\;\hat{z}$$

You can either replace $\hat{x}$, $\hat{y}$, & $\hat{z}$ w/ appropriate combinations of $\hat{r}$, $\hat{\theta}$, & $\hat{\phi}$ from the previous page, or you can work out the components one-at-a-time by doing the dot products $\vec{r}\cdot\hat{r}$, $\vec{r}\cdot\hat{\theta}$, and $\vec{r}\cdot\hat{\phi}$. Either way, you get the same answer

$$\vec{r} = r\,\hat{r}$$

- This <u>had</u> to be the answer, right? The position vector is how far a point is from the origin (r in SPC) times the direction from the origin to that point (which is what $\hat{r}$ means).
- Notice that the info about θ & ϕ is there in $\hat{r}$!

$$\hat{r} = \sin\theta\cos\phi\;\hat{x} + \sin\theta\sin\phi\;\hat{y} + \cos\theta\;\hat{z}$$

- So when you start taking derivatives of $\vec{r}$, remember that $\hat{r}$ isn't constant!

$$\frac{d}{dt}(\vec{r}) = \frac{d}{dt}\left(r\,\hat{r}(\theta,\phi)\right) = \dot{r}\,\hat{r} + r\underbrace{\frac{d}{dt}\hat{r}(\theta,\phi)}_{\text{CHAIN RULE}}$$
$$= \dot{r}\,\hat{r} + r\,\dot{\theta}\,\frac{d\hat{r}}{d\theta} + r\,\dot{\phi}\,\frac{d\hat{r}}{d\phi}$$